KEY

TO THE

'CONIC SECTIONS.'

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KEY

TO

HUNTER'S INTRODUCTION

TO THE

CONIC SECTIONS.

BY THE

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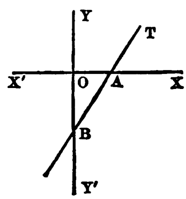
KEY

TO

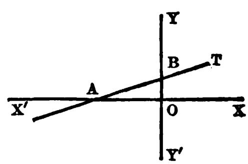
'CONIC SECTIONS.'

EXERCISES [A].

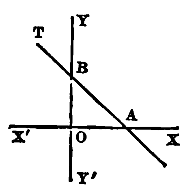
1. $3x-2y=4$. When $y=0$, then $x=1\frac{1}{3}$; and when $x=0$, then $y=-2$. Therefore, take $OA=1\frac{1}{3}$, $OB=2$. Then AB is the line required.



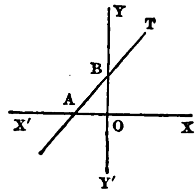
2. $x+3=4y$. When $y=0$, then $x=-3$; and when $x=0$, then $y=\frac{3}{4}$. Therefore, take $OA=3$, $OB=\frac{3}{4}$. Then AB is the line required.



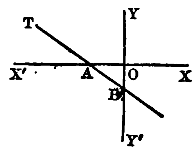
3. $4x+5y=6$. When $y=0$, then $x=1\frac{1}{2}$; and when $x=0$, then $y=1\frac{1}{5}$. Therefore, take $OA=1\frac{1}{2}$, $OB=1\frac{1}{5}$. Then AB is the line required.



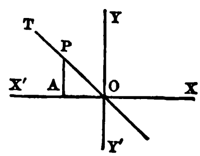
4. $\frac{1}{4}y-\frac{1}{3}x=5$; or $3y-4x=60$. When $y=0$, then $x=-15$; and when $x=0$, then $y=20$. Therefore, take OA=15, OB=20. Then AB is the line required.



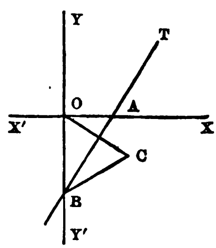
5. $3x+4y=-5$. When $y=0$, then $x=-1\frac{2}{3}$; and when $x=0$, then $y=-1\frac{1}{4}$. Therefore, take OA$=1\frac{2}{3}$, OB$=1\frac{1}{4}$. Then AB is the line required.



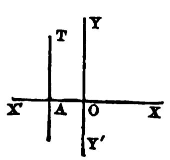
6. $x+y=0$; or $y=-x$. Here the coefficient of x is -1 which is the tangent of 135°. Therefore, make TOX=135°, and OT is the line required. For any point P being taken in OT will make AP =OA, or $y=-x$.



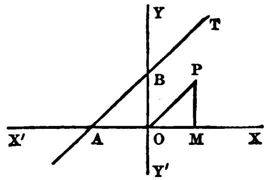
7. $x\sqrt{3}-y=8$; or $y=\sqrt{3}x-8$. Here the coefficient of x is $\sqrt{3}$, which is the tangent of 60°. Therefore, take OB=8, and on it describe the equilateral triangle OCB. The straight line BAT passing through the middle point of OC is the line required. For TAX=OAB=90° −OBA=60°.



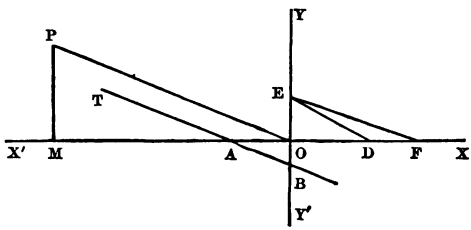
8. $x=-2$. This signifies that the abscissa of every point on the required line is $=-2$. Therefore, take OA=2, and the required line is AT, parallel to the axis of y.



9. $\frac{4}{5}x+\frac{2}{3}=\frac{1}{2}y$; or $y=\frac{8}{5}x+\frac{4}{3}$. Here the tangent of the angle which the line makes with the axis of x is $\frac{8}{5}$; therefore, take $\frac{PM}{OM}=\frac{8}{5}$, and join OP; then take $OB=1\frac{1}{3}$, and through B draw parallel to OP the required line AT.



10. $6y+x\sqrt{5}+\sqrt{10}=0$; or $y=-\frac{\sqrt{5}}{6}x-\frac{\sqrt{10}}{6}$.



Here the tangent being $-\frac{\sqrt{5}}{6}$, take $OM=6$, $PM=\sqrt{5}$, and join OP; then take $OB=\frac{1}{6}\sqrt{10}$, and through B draw parallel to OP the required line AT.

To find lines corresponding to $\sqrt{5}$ and $\sqrt{10}$, we may take $OE=1$, $OD=2$, and ED will be $=\sqrt{5}$; and if OF be taken $=3$, the line EF will be $=\sqrt{10}$.

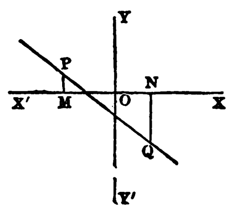
Exercises [B].

1. Take ON, QN, $=3, 5$; OM, PM, $=5, 2$; the line PQ is that required.

From art. **15** we have

$$y+5=\frac{2+5}{-5-3}(x-3);\ \text{whence}$$

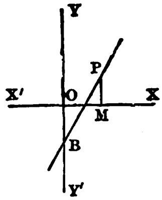
$$7x+8y=19.$$



2. Take OB=12; OM, PM, =11, 7; PB is the required line. Its equation (by art. **15**) is

$$y+12=\frac{7+12}{11-0}(x-0);$$

or $19x-11y=132$.



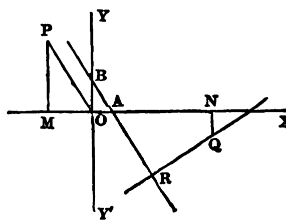
3. The given line is $y=\frac{3}{7}x-\frac{36}{7}$. And, by art. **16**, the equation to a line passing through the point (0, 0) parallel to the given line is $y-0=\frac{3}{7}(x-0)$, or $3x=7y$.

Also the equation to a line passing through the point (13, 4) parallel to the given line is

$$y-4=\frac{3}{7}(x-13);\text{ or } 3x-7y=11.$$

4. The given line is $y=-\frac{5}{3}x+\frac{4}{3}$. Take therefore $OB=\frac{4}{3}$, and $\frac{PM}{OM}=\frac{5}{3}$; join OP, and through B, parallel to OP, draw AB, which represents the given line.

Then take AN=5, QN=1, and the perpendicular QR on AB is the required line.



Now, by art. **18**, $y+1=\frac{3}{5}(x-5)$; or $3x-5y=20$.

5. The given line is $y=-\frac{7}{16}x+14$. And by art. **19** we have

$$y-30=\frac{-\frac{7}{16}\mp\sqrt{3}}{1\pm\frac{7}{16}\sqrt{3}}(x-5);$$

or, rationalising the denominators,

$$y-30=\frac{305\sqrt{3}\pm 448}{\mp 109}(x-5);$$

which gives for the required equations

$$109y+(448\pm 305\sqrt{3})x-5(1102\pm 305\sqrt{3})=0.$$

6. By art. **20**, $PQ^2=(5+7)^2+(-4-12)^2=12^2+16^2$
$=400$; $\therefore PQ=20$.

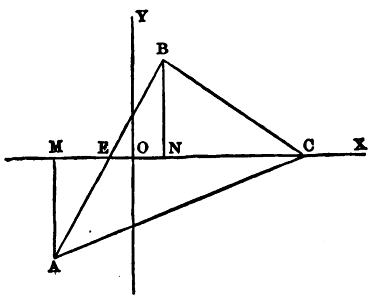
7. By art. **20**, $PQ^2=(-12-16)^2+(15-36)^2=28^2+21^2$
$=1225$; $\therefore PQ=35$.

8. Here y may be taken as the common ordinate of the point of intersection. From the 1st and 2nd equations therefore we obtain $x=-16$, $y=-15$; which values being put for x and y in the 3rd equation form the identity $-45=-32-13$. Therefore the three lines meet at one point.

9. Taking y to denote the common ordinate of the point of intersection, we have $\frac{1}{2}x+3=6x-12$; whence we obtain $x=\frac{30}{11}$, $y=\frac{48}{11}$; and, by substitution, $\frac{48}{11}=-\frac{30}{11}m+8$; which gives $m=\frac{4}{3}$.

10. From the given equations we obtain $x=-5$, $y=4$, which values being made the coordinates of a point to which a line is drawn from the origin, the equation to that line will be $4x=-5y$, or $4x+5y=0$.

11. Take OM, AM, $=21, 25$; ON, BN, $=5, 26$; OC$=46$;



and join AB, BC, CA, to form the proposed triangle.

By similar triangles we have

$$\frac{ME}{EN}=\frac{AM}{BN}; \text{ or } \frac{ME}{MN}=\frac{AM}{AM+BN}, \text{ that is, } \frac{ME}{26}=\frac{25}{51};$$

hence $ME=\frac{650}{51}$, and $EC=67-ME=\frac{2767}{51}$.

The required area is $=\frac{1}{2}EC(AM+BN)=2767\div 2=1383\frac{1}{2}$

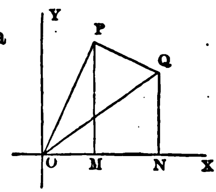
12. Take, as the coordinates of the three points, OL, AL, $=17, 20$; OM, BM, $=44, 33$; ON, CN, $=51, 8$.

The triangle is $=$ the trapezium ALMB + the trap^{m} BMNC—the trap^{m} ALNC;

$=\frac{1}{2}(AL+BM)LM+\frac{1}{2}(BM+CN)MN-\frac{1}{2}(AL+CN)LN$;

$=\frac{1}{2}(53\times 27+41\times 7-28\times 34)=383$.

13. (i.) Let the coordinates of P be x', y', and let those of Q be x'', y''. The area of OPQ is



$=$area of OPM + area of PMNQ − area of OQN;

$=\frac{1}{2}x'y'+\frac{1}{2}(x''-x')(y''+y')-\frac{1}{2}x''y''$;

$=\frac{1}{2}(x''y'-x'y'')$.

(ii.) $OP^2=OM^2+PM^2=25+144=169$;

$OQ^2=ON^2+QN^2=144+81=225$;

$PQ^2=(PM-QN)^2+MN^2=(12-9)^2+(12-5)^2=58$;

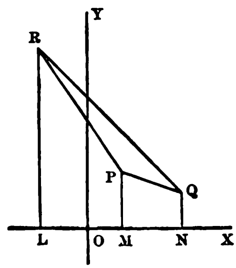
$\therefore$ OP, OQ, PQ, $=13, 15, \sqrt{58}$.

14. From 1st and 3rd equations we obtain $x=2$, $y=3$;

from 1st and 2nd we have $x=5$, $y=2$;

from 2nd and 3rd, $x=-8$, $y=28$.

These coordinates denote the points P, Q, R, respectively.



$$
\begin{array}{lr}
\text{Area of RLNQ}=\tfrac{1}{2}(28+2)\times 13= & 195 \\
\text{do. RLMP}=\tfrac{1}{2}(28+3)\times 10=155 & \\
\text{do. PMNQ}=\tfrac{1}{2}(3+2)\times 3=\ \ 7\tfrac{1}{2} & \\
 & 162\tfrac{1}{2} \\
\hline
\text{Area of PQR}= & 32\tfrac{1}{2}
\end{array}
$$

15. Take, as coordinates of the angular points, OL, AL, $=3, 4$; OM, BM, $=8, 9$; ON, CN, $=10, 6$. Also, for the coordinates of the point D take OH, DH, $=\frac{1}{2}(OL+OM)$

and $\frac{1}{2}(AL+BM)$; and for those of E take OK, EK, $=\frac{1}{2}(OM+ON)$ and $\frac{1}{2}(BM+CN)$.

Thus, $OH=\frac{1}{2}(3+8)=5\frac{1}{2}$, and $DH=\frac{1}{2}(4+9)=6\frac{1}{2}$;

also, $OK=\frac{1}{2}(8+10)=9$, and $EK=\frac{1}{2}(9+6)=7\frac{1}{2}$.

By art. **15**, the equation to DE, passing through $(5\frac{1}{2}, 6\frac{1}{2})$ and $(9, 7\frac{1}{2})$, is

$$y-6\tfrac{1}{2}=\frac{7\frac{1}{2}-6\frac{1}{2}}{9-5\frac{1}{2}}(x-5\tfrac{1}{2})\text{; or } 14y=4x+69\text{;}$$

$$\text{whence } y=\tfrac{2}{7}x+\tfrac{69}{14}.$$

The equation to AC, passing through (3, 4) and (10, 6), is

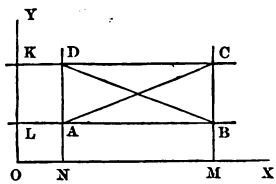
$$y-4=\frac{6-4}{10-3}(x-3)\text{; or } \quad 7y=2x+22\text{;}$$

$$\text{whence } y=\tfrac{2}{7}x+\tfrac{22}{7}.$$

Hence, the tangents of the angles which DE and AC make with the axis of x being each $=\frac{2}{7}$, the lines DE and AC are parallel.

16. Take $ON=a$, $OM=a'$, $OL=b$, $OK=b'$. DN is the line $x=a$, CM is $x=a'$.

LB is $y=b$, KC is $y=b'$.



The equation to AC, through (a, b) and (a', b'), is

$$y-b=\frac{b'-b}{a'-a}(x-a)\text{;}$$

or, $(a'-a)y-(b'-b)x-a'b+ab'=0$.

The equation to DB, through (a, b') and (a', b), is

$$y-b'=\frac{b-b'}{a'-a}(x-a)\text{; or, } (a'-a)y+(b'-b)x-a'b'+ab=0.$$

17. In OX take the abscissa $OM=12$, draw the ordinate $PM=\frac{3}{4}$ of $12=9$, and produce MP to Q making $QM=\frac{23}{12}$ of $12=23$. Join OP, OQ, and complete the parallelogram OPQR.

PQ parallel to the axis of y is the line $x=12$; OP, whose coordinates are 12, 9, is the line $4y=3x$; and OQ, whose coordinates are 12, 23, is the line $12y=23x$. The equation

to OR on the axis of y is $x=0$; also $OR=PQ=QM-PM=14$.

The equation to QR, through (0, 14) and (12, 23), is

$$y-14=\frac{23-14}{12}x; \text{ or } 4y-3x=56.$$

The equation to PR, through (0, 14) and (12, 9), is

$$y-14=\frac{9-14}{12}x; \text{ or } 5x+12y=168.$$

The area of the parallelogram $=2^{ce}$ the difference of the triangles OQM, OPM, $=OM(QM-PM)=12(23-9)=168$.

18. The equation to the diagonal AC is

$$y-3=\frac{5-3}{3-2}(x-2); \text{ or } y=2x-1.$$

We have now to find the equations to the lines AB, AD, passing through (2, 3), and making an angle of 45° with the line $y=2x-1$. In art. **19** we have the applicable formula

$$y-y'=\frac{m\pm\tan\phi}{1\mp m\tan\phi}(x-x');$$

$$\therefore\ y-3=\frac{2+1}{1-2}(x-2); \text{ or } y=-3x+9;$$

$$\text{and } y-3=\frac{2-1}{1+2}(x-2); \text{ or } 3y=x+7;$$

these are the equations to AD and AB.

Also, we have to find the equations to CB, CD, passing through (3, 5), and making an angle of 45° with the line $y=2x-1$.

$$y-5=\frac{2+1}{1-2}(x-3); \text{ or } y=-3x+14;$$

$$y-5=\frac{2-1}{1+2}(x-3); \text{ or } 3y=x+12;$$

these are the equations to CB and CD.

19. The equation to OP is $y=\frac{12}{5}x$; and that to PQ is $y=-\frac{7}{24}x+7$. Now, since every point on the bisecting line

is equidistant from OP and PQ, let (x, y) be any point on the bisecting line; its perpendicular distance from OP, by art. **21**, is

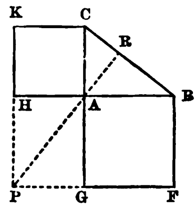
$$\frac{y-\frac{12}{5}x-0}{\sqrt{(1+\frac{144}{25})}};\text{ or }\frac{5y-12x}{13};$$

also its perpendicular distance from PQ is

$$\frac{y+\frac{7}{24}x-7}{\sqrt{(1+\frac{49}{576})}};\text{ or }\frac{24y+7x-168}{25};$$

$$\therefore \frac{5y-12x}{13}=\frac{24y+7x-168}{25};\text{ or }391x+187y=2184.$$

20. Take A as the origin, AB, AC as the axes. Let $AB=a$, $AC=b$.



Because BC cuts off intercepts a and b on the axes, its equation is

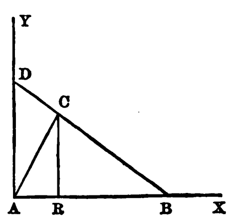
$$\frac{x}{a}+\frac{y}{b}=1;\text{ or }y=-\frac{b}{a}x+b. \quad (1)$$

Again, since AK and AF are squares, P is the point $(-b, -a)$; and the equation to PA passing through the origin is

$$y=\frac{a}{b}x. \quad (2)$$

Accordingly (1) and (2), having reciprocal coefficients of x with opposite signs, are perpendicular to each other.

21. Here the given equations supply the respective tangents of CAX and CBX, viz. m' and m; $\therefore$ tan CBA$=-m$.



Tan ACB$=$tan (CBX$-$CAX)

$$=\frac{m-m'}{1+mm'}.$$

The sides may be found as follows:

AD being $=c$, and $\frac{AB}{AD}=\frac{BR}{CR}$

$=-\frac{1}{m}$, we have the magnitude of $AB=\frac{c}{m}$.

Then $\frac{CR}{AR}=m'$; $\frac{BR}{CR}=-\frac{1}{m}$; $\therefore \frac{AR}{BR}=\frac{m}{-m'}$; $\frac{AR}{AB}=\frac{m}{m-m'}$;

$$\text{hence, } AR=\frac{c}{m-m'},\ CR=\frac{cm'}{m-m'},\ BR=\frac{cm'}{m(m-m')}.$$

$$AC^2=AR^2+CR^2=\frac{c^2}{(m-m')^2}(1+m'^2);$$

$$\text{or } AC=\frac{c}{m-m'}\sqrt{(1+m'^2)};$$

$$BC^2=BR^2+CR^2=\frac{c^2m'^2}{(m-m')^2}\cdot\frac{1+m^2}{m^2};$$

$$\text{or } BC=\frac{cm'}{m(m-m')}\sqrt{(1+m^2)}.$$

Exercises [C].

1. (i.) Completing squares with x^2-6x and y^2-8y, we have

$$(x-3)^2+(y-4)^2=24+9+16=49.$$

Therefore the radius $=\sqrt{49}=7$; and the coordinates of the centre are 3 and 4.

(ii.) Completing squares with x^2+2x and y^2+4y, we have

$$(x+1)^2+(y+2)^2=4+1+4=9.$$

Therefore the radius $=\sqrt{9}=3$; and the coordinates of the centre are -1 and -2.

(iii.) Completing squares with x^2-6x and y^2-18y, we have

$$(x-3)^2+(y-9)^2=80=16\times 5.$$

Therefore the radius $=4\sqrt{5}$; and the coordinates of the centre are 3 and 9.

(iv.) Completing squares with x^2+x and y^2-2y, we have

$$(x+\tfrac{1}{2})^2+(y-1)^2=5.$$

Therefore the radius $=\sqrt{5}$; and the coordinates of the *centre* $-\frac{1}{2}$ *and* 1.

2. (i.) The second equation gives $y^2=x^2-4x+4$; therefore $x^2-4x+4=34-x^2$; whence $x=5$ or -3; $y=3$ or -5; so that the points of intersection are $(5, 3)$ and $(-3, -5)$.

(ii.) The 2nd equation gives $y^2=4x^2+4x+1$; therefore $4x^2+4x+1=34-x^2$; whence $x=2\frac{1}{5}$ or -3; $y=-5\frac{2}{5}$ or 5; so that the points of intersection are $(2\frac{1}{5}, -5\frac{2}{5})$ and $(-3, 5)$.

3. (i.) The 2nd equation gives $16y^2=9x^2-30x+25$; therefore $9x^2-30x+25=16-16x^2$; or $25x^2-30x+9=0$; $\therefore 5x-3=0$; whence $x=\frac{3}{5}$; $y=\frac{1}{4}(3x-5)=-\frac{4}{5}$; so that the line meets the circle at only one point, being a tangent to it at the point $(\frac{3}{5}, -\frac{4}{5})$.

(ii.) The second equation gives $16y^2=25x^2+50x+25$; therefore $25x^2+50x+25=16-16x^2$; whence $x=-1$ or $-\frac{9}{41}$; $y=-\frac{1}{4}(5x+5)=0$ or $-\frac{40}{41}$; so that the line meets the circle at the points $(-1, 0)$ and $(-\frac{9}{41}, -\frac{40}{41})$, the former of these points being an extremity of the diameter in the axis of x.

4. (i.) From the two given equations we obtain

$$16y^2=9x^2-192x+1024=-16x^2+384x+160y;$$

whence, substituting for $160y$ its value in terms of x, viz.

$$120x-1280, \text{ we have } 25x^2-696x=-2304;$$

$\therefore x=24$ or $\frac{96}{25}$; $y=\frac{1}{4}(3x-32)=10$ or $-\frac{128}{25}$; so that the points of intersection (the origin being on the circumference) are $(24, 10)$ and $(3{\cdot}84, -5{\cdot}12)$, the former of these being an extremity of the diameter through the origin.

(ii.) From the 1st equation $y=\frac{3}{4}x-8$, and $2y^2=1\frac{1}{8}x^2-24x+128$; hence, substituting these values in the 2nd equation, we have $x^2-10x=-24$; $\therefore x=6$ or 4, $y=-3\frac{1}{2}$ or -5; so that the points of intersection are $(6, -3\frac{1}{2})$ and $(4, -5)$.

5. Let (x', y') be the point on the circle at which the tangent is drawn: The equation to the tangent (see art **28**) is

$$(x-a)(x'-a)+(y-b)(y'-b)=c^2;$$

which gives $y-b=-\dfrac{x'-a}{y'-b}(x-a)+\dfrac{c^2}{y'-b}$;

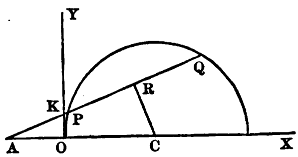
or, expressing $\dfrac{c^2}{y'-b}$ in terms of the tangent $-\dfrac{x'-a}{y'-b}$, we have, by art. **29**,

$$y-b=-\frac{x'-a}{y'-b}(x-a)\pm c\sqrt{\frac{(x'-a)^2}{(y'-b)^2}+1}.$$

Now, in order that this line may be parallel to the line $y=mx+n$, we must have $-\dfrac{x'-a}{y'-b}=m$; hence the required equation is

$$y-b=m(x-a)\pm c\sqrt{(1+m^2)}.$$

6. The equation to the circle may be written



$$(x-20)^2+y^2=400;$$

so that the radius is 20. The equation to the straight line, AQ, may be written

$$\frac{x}{-11\frac{1}{5}}+\frac{y}{4\frac{2}{3}}=1; \text{ so that } OA=11\tfrac{1}{5},\ OK=4\tfrac{2}{3};$$

$$\therefore AK=\sqrt{(OA^2+OK^2)}=\tfrac{182}{15}.$$

Now, by art. **31**, the perpendicular CR from the centre bisects the chord PQ; and by similar triangles we have

$$\frac{CR}{AC}=\frac{OK}{AK}, \text{ or } CR=\tfrac{14}{3}\times\tfrac{15}{182}\times\tfrac{156}{5}=12;$$

hence $\quad PR=\sqrt{(OC^2-CR^2)}=16;\ \therefore PQ=32.$

7. Squaring the equation to the line gives $y^2=\dfrac{b^2}{a^2}x^2-\dfrac{2b^2}{a}x+b^2$; whence by substitution in the equation to the circle we have

$$\frac{a^2+b^2}{a^2}x^2-\frac{2b^2}{a}x+b^2-c^2=0,$$

$$\text{or,}\quad x^2-\frac{2ab^2}{a^2+b^2}x+\frac{a^2(b^2-c^2)}{a^2+b^2}=0.$$

This quadratic will have two roots, except when the third term is the square of half the coefficient of the second term; that is, when

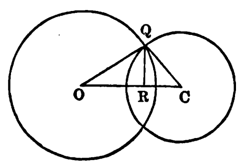
$$\frac{a^2(b^2-c^2)}{a^2+b^2}=\left(\frac{ab^2}{a^2+b^2}\right)^2; \text{ or when } (b^2-c^2)(a^2+b^2)=b^4;$$

$$\text{or when } b^2c^2+a^2c^2=a^2b^2;$$

which, by dividing each term by $a^2b^2c^2$, becomes

$$\frac{1}{a^2}+\frac{1}{b^2}=\frac{1}{c^2}.$$

8. Let O, the centre of the circle whose radius is c, be the origin of coordinates: The equation to that circle is $x^2+y^2=c^2$. Now $CR^2+QR^2=CQ^2$, or $(a-x)^2+y^2=c'^2$; or $a^2-2ax+x^2+y^2=c'^2$; that is, $a^2-2ax+c^2=c'^2$.



$$\therefore x=\frac{1}{2a}(a^2+c^2-c'^2).$$

Again; $y^2=c^2-x^2=c^2-\frac{1}{4a^2}(a^2+c^2-c'^2)^2$;

$$\therefore y=\pm\frac{1}{2a}\sqrt{4a^2c^2-(a^2+c^2-c'^2)^2}$$

$$=\pm\frac{1}{2a}\sqrt{(2ac+a^2+c^2-c'^2)(2ac-a^2-c^2+c'^2)}$$

$$=\pm\frac{1}{2a}\sqrt{\{(a+c)^2-c'^2\}\{c'^2-(a-c)^2\}}$$

$$\text{or } y=\pm\frac{1}{2a}\sqrt{(a+c+c')(a+c-c')(a-c+c')(c-a+c')}.$$

9. The general equation to the circle being

$$x^2+y^2+Ax+By+C=0,$$

and B being in the present case $=0$, the centre is on the axis of x at the distance $\frac{1}{2}A=\frac{m^2+1}{m^2-1}a$ from the origin. The

$$\text{radius } =\sqrt{\tfrac{1}{4}A^2-a^2}=a\sqrt{\left(\frac{m^2+1}{m^2-1}\right)^2-1}$$

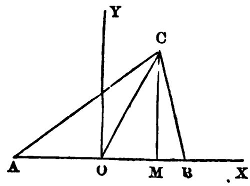
$$=a\sqrt{\frac{2m^2}{m^2-1}\cdot\frac{2}{m^2-1}}=\frac{2m}{m^2-1}a.$$

To determine the points at which the circle cuts the axis of x, make $y=0$ in the general equation, and the values of x will then be found

$$=-\tfrac{1}{2}A\pm\sqrt{\tfrac{1}{4}A^2-C}\,;\ =\frac{m^2+1}{m^2-1}a\pm\frac{2m}{m^2-1}a,$$

$$=\frac{m^2\pm 2m+1}{m^2-1}a,\ =\frac{m+1}{m-1}a \text{ and } \frac{m-1}{m+1}a.$$

10. Take O, the middle point of the base AB, as origin, $AO=OB=a$; and let (x, y) be the vertex C.



$AC^2=CM^2+AM^2=y^2+(a+x)^2$;
$BC^2=CM^2+BM^2=y^2+(a-x)^2$;
whence, by addition,

$$2(y^2+x^2+a^2)=AC^2+BC^2,$$

that is,

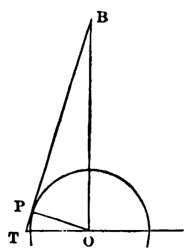
$$2(OC^2+OA^2)=AC^2+BC^2.$$

Again; putting s for AC^2+BC^2,

$$y^2+x^2+a^2=\tfrac{1}{2}s\,;\text{ or } x^2+y^2=\tfrac{1}{2}s-a^2,$$

the equation to a circle whose radius $=\sqrt{\tfrac{1}{2}s-a^2}$.

11. Take the intercept $OB=5$, $OT=\frac{1}{3}$ of 5. TB is the line touching the required circle.



$$TB^2=OB^2+OT^2=\tfrac{250}{9}\,;$$

$$\therefore TB=\tfrac{1}{3}\sqrt{250}.$$

Now, $OP\cdot TB=OB\cdot OT$, that is,

$$OP\times\tfrac{1}{3}\sqrt{250}=5\times\tfrac{5}{3}\,;$$

$\therefore OP^2=625\div 250=2\frac{1}{2}$; and the equation to the circle is, therefore,

$$x^2+y^2=2\tfrac{1}{2}.$$

12. From the given equations we obtain $x=\frac{1}{2}a$, and $y=\frac{1}{2}\sqrt{4c^2-a^2}$; that is, the coordinates of the centre of the required circle are $\frac{1}{2}a$ and 0; and those of one of the points of intersection of the two given circles are $\frac{1}{2}a$ and $\frac{1}{2}\sqrt{4c^2-a^2}$; therefore, the square of the radius of the required circle is $(\frac{1}{2}\sqrt{4c^2-a^2})^2=c^2-\frac{1}{4}a^2$.

Hence the required equation is

$(x-\frac{1}{2}a)^2+(y-0)^2=c^2-\frac{1}{4}a^2$; or, $x^2+y^2-ax=c^2-\frac{1}{2}a^2$.

13. The two centres, P and Q, are the points (a, b) and (b, a), and, by art. **20**, the length of the line joining these points is determined by

$$PQ^2=(b-a)^2+(a-b)^2=2(a-b)^2;$$

hence the distance between the centres is $\sqrt{2(a-b)^2}$; and as the radii are equal, we have $c=\pm\frac{1}{2}\sqrt{2}(a-b)$.

14. The equation $x^2+y^2-2cx=0$ denotes that the origin is the extremity of a diameter in the axis of x.

The condition of intersection of the circle with the line $y=mx$, or $y^2=m^2x^2$, is $2cx-x^2=m^2x^2$; whence $x=\dfrac{2c}{1+m^2}$, $y=\dfrac{2mc}{1+m^2}$; and the square of the line $=\dfrac{4c^2+4m^2c^2}{(1+m^2)^2}=\dfrac{4c^2}{1+m^2}$;

$\therefore$ square of radius of required circle $=\dfrac{c^2}{1+m^2}$.

The coordinates, then, of the centre of the required circle being $\dfrac{c}{1+m^2}$ and $\dfrac{mc}{1+m^2}$, we have, for the required equation,

$$\left(x-\frac{c}{1+m^2}\right)^2+\left(y-\frac{mc}{1+m^2}\right)^2=\frac{c^2}{1+m^2};$$

$$\text{or } x^2+y^2-\frac{2c}{1+m^2}\,x-\frac{2mc}{1+m^2}\,y+\frac{c^2}{1+m^2}=\frac{c^2}{1+m^2};$$

$$\text{that is, } x^2+y^2-\frac{2c}{1+m^2}\,(x-my)=0.$$

Exercises [D].

1. (i.) Dividing by c^2, we have

$$\frac{x^2}{\frac{1}{3}c^2}+\frac{y^2}{\frac{1}{4}c^2}=1;\ \therefore a^2=\tfrac{1}{3}c^2, \text{ and } b^2=\tfrac{1}{4}c^2;$$

$$\therefore \frac{b^2}{a^2}=\tfrac{3}{4};\ \text{and } 1-\frac{b^2}{a^2}=e^2=\tfrac{1}{4};\ \therefore e=\tfrac{1}{2}.$$

(ii.) Dividing by c'^2, we have

$$\frac{x^2}{\frac{1}{3}c'^2}+\frac{y^2}{\frac{1}{5}c'^2}=1\,;\ \therefore\ a^2=\tfrac{1}{3}c'^2,\text{ and }b^2=\tfrac{1}{5}c'^2\,;$$

$$\therefore\ \frac{b^2}{a^2}=\tfrac{3}{5}\,;\text{ and }1-\frac{b^2}{a^2}=e^2=\tfrac{2}{5}=\tfrac{10}{25}\,;\ \therefore\ e=\tfrac{1}{5}\sqrt{10}.$$

(iii.) Dividing by 780^2 shews $a^2=156^2$, and $b^2=60^2$;

$$\therefore\ \frac{b^2}{a^2}=\tfrac{25}{169}\,;\text{ and }1-\frac{b^2}{a^2}=e^2=\tfrac{144}{169}\,;\ \therefore\ e=\tfrac{12}{13}.$$

2. The coordinates of the points A′ and B are $-a$, 0, and 0, b; those of C and F are 0, 0, and ae, $\frac{b^2}{a}$; hence the equations to the lines A′B and CF are, respectively,

$$y=\frac{b}{a}\,x+b,\text{ and }y=\frac{b^2}{a^2}\cdot\frac{1}{e}\,x\,;$$

and that these lines may be parallel they must have the same inclination to the axis of x, or the tangents $\frac{b}{a}$ and $\frac{b^2}{a^2}\cdot\frac{1}{e}$ must be equal;

that is, $\sqrt{1-e^2}=\frac{1}{e}\,(1-e^2)$, or $1=\frac{1}{e^2}(1-e^2)$; $\therefore\ e=\frac{1}{2}\sqrt{2}$.

3. $a=\frac{1}{2}(4+6)=5$; $SC=ae=4$; $SH=8$;

$SM^2-MH^2=SP^2-HP^2$; whence $SM-MH=2\frac{1}{2}$;

$\therefore\ SM=5\frac{1}{4}$; $x'=CM=SM-SC=1\frac{1}{4}$.

$PM^2=(a^2-x'^2)(1-e^2)=\frac{9}{16}$ of 15; $\therefore\ y'=PM=\frac{3}{4}\sqrt{15}$.

4. The equation to the normal at the point (x', y') is

$$y-y'=\frac{a^2y'}{b^2x'}\,(x-x').$$

In the present case, we have $x'=CH=ae$, and $y'=HF=\frac{b^2}{a}$; and therefore $\frac{a^2y'}{b^2x'}=\frac{ab^2}{ab^2e}=\frac{1}{e}$; hence the equation *becomes*

$$y-\frac{b^2}{a}=\frac{1}{e}\,(x-ae)=\frac{x}{e}-a,$$

$$\text{or } y+\frac{a^2-b^2}{a}=\frac{x}{e}\,;$$

but $b^2=a^2-a^2e^2$, or $a^2-b^2=a^2e^2$; therefore, the equation to the normal at F is $y+ae^2=\frac{x}{e}$.

5. $y'^2=(a^2-x'^2)(1-e^2)$;

$$\text{that is, } \frac{49\times 119}{32}=(a^2-\tfrac{729}{32})\times\tfrac{49}{81}\,;$$

or, $119=(32a^2-729)\times\frac{1}{81}$; or $a^2=324$; $\therefore\ a=18$.

$SP=a+ex=18+\frac{4}{9}\sqrt{2}\times\frac{27}{8}\sqrt{2}=18+3=21$;

$HP=a-ex \qquad\qquad =18-3=15$.

6. The equation $2x^2+3y^2=18$ may be written

$$\frac{x^2}{9}+\frac{y^2}{6}=1, \text{ where } 9=a^2,\ 6=b^2.$$

Now, from art. **46**, the equation to the tangent is

$$y=mx\pm\sqrt{a^2m^2+b^2}\,;$$

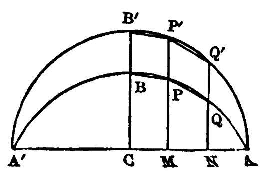
and we have here $m=\tan 30°=\frac{1}{3}\sqrt{3}$;

$$\therefore\ y=\tfrac{1}{3}\sqrt{3}x\pm\sqrt{3+6}\,;\ \text{or } 3y=x\sqrt{3}\pm 9.$$

7. Given $a^2=2b^2$; $\therefore\ a^2-b^2=b^2=\frac{1}{2}a^2$; that is, $a^2e^2=\frac{1}{2}a^2$;
$\therefore\ 4a^2e^2=2a^2$; that is, $SH^2=SB^2+HB^2$.

8. (*See* fig. in art. **49.**) Since $\sin VCT=\cos VTC=\cos\phi$, we have, from art. **51**, the perpendicular to the tangent, from the centre as origin, $=a(1-e^2\cos^2\phi)^{\frac{1}{2}}$; and it is an arithmetical mean between HR and SR′. Suppose CZ drawn perpendicular to SR′, then $SZ=SC\sin SCZ=ae\sin\phi$; $\therefore$ SR′ and $HR=a\{e\sin\phi\pm(1-e^2\cos^2\phi)^{\frac{1}{2}}\}$.

9. Let BC, PM, QN be ordinates of an ellipse, and let them be produced to B′, P′, Q′, points in the circumference of a circle described on the transverse axis.



By art. **49** we have

$$\frac{BC}{B'C}=\frac{PM}{P'M}=\frac{QN}{Q'N}=\frac{b}{a};$$

$$\therefore \frac{BC+PM}{B'C+P'M}=\frac{PM+QN}{P'M+Q'N}=\frac{b}{a}.$$

Now, the areas of the trapeziums BCMP and B'CMP' are $\frac{1}{2}$CM(BC+PM) and $\frac{1}{2}$CM(B'C+P'M), and are therefore as b to a; and the same ratio holds for the trapeziums PN and P'N, and for every pair of trapeziums similarly situated; so that a polygon of an indefinite number of sides inscribed in the ellipse will be to the corresponding polygon in the circle as b to a; that is, the area of the ellipse is to that of the circle as b to a.

Accordingly, since the area of the circle described on the transverse axis of the ellipse is πa^2, we have

$$a : b :: \pi a^2 : \pi ab, \text{ the area of the ellipse.}$$

Hence, if r be the radius of a circle equal in area to an ellipse whose semi-axes are a and b, we have $\pi ab=\pi r^2$; that is, $ab=r^2$, or the radius is a mean proportional between the semi-axes.

10. In the given ellipse put c^2 for a^2, as the a here has not the same import as that in the general equation; then $a^2=9c^2$, and $b^2=4c^2$; $\therefore$ $a^2-b^2=a^2e^2$ becomes $5c^2=9e^2c^2$; hence $e=\frac{1}{3}\sqrt{5}$.

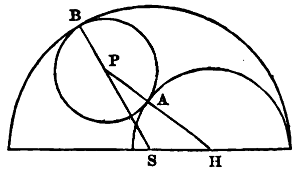
The equation to the tangent at the point (x', y') is $y=-\frac{b^2x'}{a^2y'}x+\frac{b^2}{y'}$; which, for $x'=ae, y'=\frac{b^2}{a}$, becomes $y=-ex+a$. Now, instead of a resume $\sqrt{9a^2}$, or $3a$, from the given form of equation, and the required equation will be

$$y=-\tfrac{1}{3}\sqrt{5}x+3a;$$

$$\text{or, } 3y+\sqrt{5}x-9a=0.$$

From the first of these forms we see that the required intercept on the axis of y is $3a$; and that on the axis of x is

$3a\div\frac{1}{3}\sqrt{5}=9a\div\sqrt{5}=\frac{9a}{5}\sqrt{5}.$



11. Let S and H be the centres of the given circles, P the centre of an intervening circle touching the given circles at A and B.

Join SP, HP, which will pass through the points of contact.

$$HP+SP=HA+AP+SB-BP\text{; but } AP=BP\text{;}$$

$\therefore$ $HP+SP=HA+SB$, the sum of the given radii, and therefore constant.

And the locus of a point, the sum of whose distances from two given points is constant, is an ellipse. (*See* the 4th of the worked Examples.)

12. If x', y' were the coordinates of the point P: the equation to A′P passing through $(-a, 0)$ and (x', y') would be

$$y=\frac{y'}{x'+a}(x+a)\,;$$

and the equation to AP through $(a, 0)$ and (x', y') would be

$$y=\frac{y'}{x'-a}(x-a).$$

Hence, by art. **17**, the tangent of the angle of intersection of A′P and AP is

$$=\left(\frac{y}{x-a}-\frac{y}{x+a}\right)\div\left(1+\frac{y^2}{x^2-a^2}\right)=\frac{2ay}{x^2-a^2+y^2}.$$

But, as the lines are to intersect on the ellipse, we have

$$a^2y^2+b^2x^2=a^2b^2\text{; or, } x^2-a^2=-\frac{a^2}{b^2}y^2\text{;}$$

whence, by substitution, the tangent of the angle of intersection becomes $=2ay\div\left(1-\frac{a^2}{b^2}\right)y^2$

$$=\frac{-2ab^2}{(a^2-b^2)y}=\frac{-2ab^2}{a^2e^2y}=\frac{-2b^2}{ae^2y}.$$

EXERCISES [E].

1. Here $e^2-1=4-1=3=\frac{b^2}{a^2}$; $\therefore b^2=3a^2$; and the equation to the hyperbola, viz. $y^2=\frac{b^2}{a^2}x^2-b^2$, becomes

$$y^2=3(x^2-a^2).$$

2. Here we have $3x^2-2y^2=-6n^2$; or, $\frac{x^2}{-2n^2}-\frac{y^2}{-3n^2}=1$;

$$\therefore a^2 : b^2 :: 2n^2 : 3n^2 :: 2 : 3;$$

$$e^2-1=\frac{b^2}{a^2}=\frac{3}{2}; \text{ or } e^2=\frac{5}{2}=\frac{10}{4}; \therefore e=\tfrac{1}{2}\sqrt{10}.$$

And, since $\frac{b^2}{a^2}=\frac{3}{2}$, $\therefore \frac{2b^2}{a}=3a$, the latus rectum.

3. In the ellipse the coordinates of A are $x'=a$, $y'=0$; for that point, therefore, the equation to the tangent,

$$\text{viz. } a^2yy'+b^2xx'=a^2b^2,$$

becomes $\qquad x=a,$

which is the equation to a line through A parallel to CY.

In the hyperbola, the coordinates of A are $x'=a$, $y'=0$; for that point, therefore, the equation to the tangent,

$$\text{viz. } a^2yy'-b^2xx'=-a^2b^2,$$

again becomes $\qquad x=a.$

4. The equation to the tangent at P is

$$a^2yy'-b^2xx'=-a^2b^2.$$

The intercept CT, made by the tangent TP, is found from the equation to the tangent by putting $y=0$; whence $x=\frac{a^2}{x'}=\text{CT}$; $\therefore \frac{\text{CT}}{\text{CA}}=\frac{a}{x'}=\frac{\text{CA}}{\text{CM}}=\frac{\text{CE}}{\text{CP}}$;

$\therefore$ CTE and CAP are similar triangles.

5. From $\frac{y}{m}-\frac{x}{n}=1$ we have $y=\frac{m}{n}x+m$; (1)

and the tangent to the hyperbola being of the form

$$y=m'x\pm\sqrt{a^2m'^2-b^2}, \qquad (2)$$

m', the tangent of inclination to the transverse axis, is $=\frac{m}{n}$; $\therefore$ (2) becomes

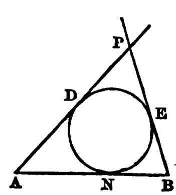
$$y=\frac{m}{n}x\pm\sqrt{a^2\frac{m^2}{n^2}-b^2};$$

hence, $$\frac{m}{n}x+m=\frac{m}{n}x\pm\sqrt{a^2\frac{m^2}{n^2}-b^2},$$

$$\text{or, } a^2\frac{m^2}{n^2}-b^2=m^2; \text{ or, } \frac{a^2}{n^2}-\frac{b^2}{m^2}=1.$$

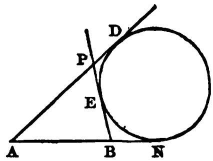
6. The equation to a line touching the ellipse (*see* art. 46) is
$$y=mx+\sqrt{a^2m^2+b^2}; \qquad (1)$$
that to a line touching the hyperbola is of the form
$$y=m'x+\sqrt{a'^2m'^2-b'^2}. \qquad (2)$$
In order that (2) may be at right angles to (1) we must have $m'=-\frac{1}{m}$; hence (2) becomes
$$y=-\frac{1}{m}x+\frac{\sqrt{a'^2-b'^2m^2}}{m},$$
$$\text{or } my=-x+\sqrt{a'^2-b'^2m^2}. \qquad (3)$$
Squaring (1) and (3) and then adding the results, we have
$$1+m^2(x^2+y^2)=a'^2+b^2+(a^2-b'^2)m^2; \qquad (4)$$
but, since the curves have the same foci and centre,
$$SC^2=a^2-b^2=a'^2+b'^2; \therefore a^2-b'^2=a'^2+b^2.$$
Thus (4) becomes
$$(1+m^2)(x^2+y^2)=(a'^2+b^2)(1+m^2),$$
$$\text{or, } x^2+y^2=a'^2+b^2;$$
which is the equation to a circle passing through the four points of intersection of the curves.

7. If N, the fixed point in AB, is between A and B, and AN$>$NB, we have
AP $-$ PB $=$ AD $-$ BE $=$ AN $-$ NB, a constant difference; hence, in this case, the locus of P is an hyperbola of which A and B are the foci.



If, however, AN=NB, the locus will be the perpendicular to AB at N.

But if the point N be on AB produced, then AP + PB = AD + EB =AN + BN, a constant sum; so that in this case the locus of P is an ellipse.



8. Magnitude of cos PSH$=\frac{SM}{SP}=\frac{x+ae}{ex+a}$;

$$\tan^2 \tfrac{1}{2}PSH=\frac{\sin^2 \frac{1}{2}PSH}{\cos^2 \frac{1}{2}PSH}=\frac{1-\cos PSH}{1+\cos PSH}$$

$$=\frac{1-\dfrac{x+ae}{ex+a}}{1+\dfrac{x+ae}{ex+a}}=\frac{(e-1)(x-a)}{(e+1)(x+a)}.$$

Magnitude of cos PHS$=\frac{HM}{HP}=\frac{x-ae}{ex-a}$;

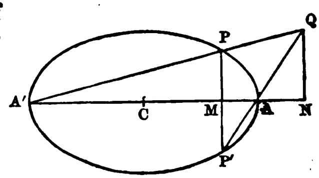
$$\tan^2 \tfrac{1}{2}PHS=\frac{1-\dfrac{x-ae}{ex-a}}{1+\dfrac{x-ae}{ex-a}}=\frac{(e-1)(x+a)}{(e+1)(x-a)}.$$

$$\therefore \tan \tfrac{1}{2}PSH \tan \tfrac{1}{2}PHS=\frac{e-1}{e+1}.$$

9. Let the coordinates of Q be x and y; let those of P be x' and y'.

By similar triangles,

$\frac{QN}{AN}=\frac{P'M}{AM}$, or $\frac{y}{x-a}=\frac{y'}{a-x'}$;



$$\frac{QN}{A'N}=\frac{PM}{A'M}, \text{ or } \frac{y}{x+a}=\frac{y'}{a+x'};$$

$\therefore$ by mult^n $\frac{y^2}{a^2-x^2}=\frac{y'^2}{a^2-x'^2}$. But the equation to the

ellipse is $y'^2=\frac{b^2}{a^2}(a^2-x'^2)$, or $\frac{y'^2}{a^2-x'^2}=\frac{b^2}{a^2}$;

$\therefore \frac{y^2}{x^2-a^2}=\frac{b^2}{a^2}$, or, $y^2=\frac{b^2}{a^2}(x^2-a^2)$, the equation to an hyperbola having the same axes as the ellipse.

10. The centre of the hyperbola being origin, the equation to the circle is

$$(x-ae)^2+(y-0)^2=b^2=a^2(e^2-1). \qquad (1)$$

The equation to the asymptotes is $y=\pm\frac{b}{a}x$;

$$\therefore y^2=\frac{b^2}{a^2}x^2=(e^2-1)x^2;$$

hence, by substitution in (1), that the circle may meet the asymptotes,

$$(x-ae)^2+(e^2-1)x^2=a^2(e^2-1);$$

$$\text{or } e^2x^2-2aex+a^2=0; \text{ or, } (ex-a)^2=0;$$

$$\therefore x=\frac{a}{e}.$$

Accordingly, the circle touches each of the asymptotes at a point of which the abscissa is $\frac{a}{e}$, that is, at the point where the directrix meets them.

Exercises [F].

1. Since LS=2AS, the equation to AL is $y=2x$. The equation to a tangent is $yy'=2a(x+x')$; and in the present case $y'=2a$, $x=a$; $\therefore$ the equation to the tangent at L is $2ay=2a(x+a)$, or $y=x+a$; hence, by art. **17**, the angle of intersection of the lines

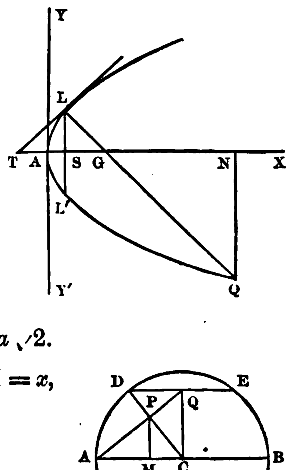
$$y=2x \text{ and } y=x+a,$$

is that whose tangent is $\frac{2-1}{1+2\times 1}$, or $\frac{1}{3}$.

2. Since ST=SP, and the angle PTS=60°, the triangle PTS is equilateral, and PM bisects TS; and since AM=AT, $\therefore$ AT=$\frac{1}{3}$AS=$\frac{1}{3}a$; hence TS=TP=$\frac{4}{3}a$.

3. Suppose the circle to meet AX at N, then AN, at right angles to LS, is the diameter, and LS is a mean proportional between AS and SN, or $\frac{\text{SN}}{\text{LS}}=\frac{\text{LS}}{\text{AS}}$, that is, $\frac{\text{SN}}{2a}=2$, or SN$=4a$; $\therefore$ AN$=5a$.

4. *See* fig. in art. **80**. When the angle PTS=30°, then the angle PSG=60°; and the triangle SPG is equilateral; also SM=MG=$2a$; $\therefore$ SP=SG=$4a$=the latus rectum.



5. From art. **80** we have

SG=SL=$2a$; $\therefore$ AG=$3a$,
and LG=$2a\sqrt{2}$.

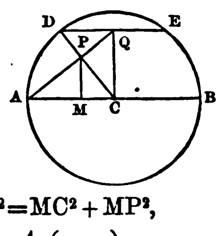
Let the coordinates of Q be AN=x, QN=y; then, since GN=NQ, we have $y=x-3a$; also, since Q is on a parabola, $y^2=4ax$; $\therefore$ by substitution,

$$x^2-6ax+9a^2=4ax,$$

whence $x=9a$; $\therefore$ GN$=6a$;

GQ$=6a\sqrt{2}$;

hence LGQ$=2a\sqrt{2}+6a\sqrt{2}=8a\sqrt{2}$.



6. Let AC = CD = $2a$, AM = x, PM=y. By similar triangles,

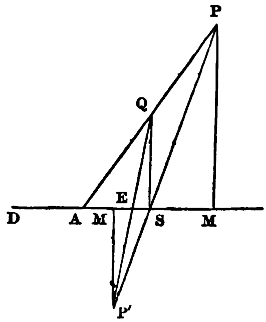
$$\frac{\text{QC}}{\text{AC}} \text{ or } \frac{\text{QC}}{2a}=\frac{y}{x}\text{; also,}$$

$$\frac{\text{PM}}{\text{PC}}=\frac{\text{QC}}{\text{CD}}=\frac{\text{QC}}{\text{AC}}=\frac{y}{x}\text{; but PM}=y,$$

$$\therefore \text{PC}=x=\text{AM};\ \therefore \text{AM}^2=\text{PC}^2=\text{MC}^2+\text{MP}^2,$$

that is, $x^2=(2a-x)^2+y^2$; hence $y^2=4a(x-a)$, which is the equation to a parabola of which a tangent to the circle at A is the directrix, and the centre of the circle is the focus.

7. Let x, y be the coordinates of P, x', y' those of P′.



By similar triangles, $\frac{SM}{SM'}=\frac{SP}{SP'}=\frac{DM}{DM'}$; $\therefore \frac{SM}{DM}=\frac{SM'}{DM'}$,

that is, $$\frac{x-a}{x+a}=\frac{a-x'}{a+x'};\ \therefore\ \frac{a}{x}=\frac{x'}{a}. \qquad (1)$$

Now, $y^2=4ax$, and $y'^2=4ax'$; $\therefore x=\frac{y^2}{4a}$, and $x'=\frac{y'^2}{4a}$;

hence, by substitution in (1), we obtain $\frac{4a}{y}=\frac{y'}{a}$.

Again, $\frac{SQ}{AS}=\frac{PM}{AM}$;

$$\text{or, } \frac{SQ}{a}=\frac{y}{x}=\frac{4a}{y}=\frac{y'}{a}=\frac{P'M'}{a};\ \therefore SQ=P'M';$$

but the triangles QSE and PM′E are similar; $\therefore$ QE=EP′.

8. Let AM=x, PM=y; draw QN′ perpendicular to DS, and let SN′=h, QN′=k. By similar triangles,

$$\frac{SN'}{QN'}=\frac{SM}{PM},\ \text{or}\ \frac{h}{k}=\frac{a-x}{y},$$

$$\frac{DS}{SN}=\frac{DM}{PM},\ \text{or}\ \frac{2a}{k}=\frac{a+x}{y};$$

$$\therefore\ \frac{2a+h}{k}=\frac{2a}{y},\ \text{and}\ \frac{2a-h}{k}=\frac{2x}{y};$$

hence, by multiplication, $\frac{4a^2-h^2}{k^2}=\frac{4ax}{y^2}=1$,

$$\text{or, } h^2+k^2=4a^2,$$

which is the equation to a circle whose centre is S, and radius $=2a=$ DS.

9. Let $y^2=4ax$ be the equation to the parabola. The equation to the tangent, in terms of the tangent of the angle which the line makes with the axis, is

$$y'=mx'+\frac{a}{m}, \text{ or } m^2-\frac{y'}{x'}m+\frac{a}{x'}=0;$$

solving this quadratic, we obtain

$$2mx'=y'\pm\sqrt{y'^2-4ax'};$$

$$\text{but } x'=\frac{y'}{m}-\frac{a}{m^2}, \text{ or } 2mx'=2y'-\frac{2a}{m};$$

$$\therefore 2y'-\frac{2a}{m}=y'\pm\sqrt{y'^2-4ax'};$$

$$\text{hence } \frac{1}{m}=\frac{y'}{2a}\mp\frac{1}{2a}\sqrt{y'^2-4ax'}.$$

The difference of these two cotangents is

$$\frac{1}{a}\sqrt{y'^2-4ax}=d, \text{ a constant};$$

$$\therefore y^2-4ax=a^2d^2, \text{ or } y^2=4a(x+\tfrac{1}{4}ad^2),$$

which is the eqn. to a parabola whose latus rectum is $4a$, and vertex at a distance $=\frac{1}{4}ad^2$ from the origin.

EXERCISES [G].

1. $\frac{\sin^2\theta}{\cos^2\theta}=\frac{9}{1}$; $\therefore \frac{\sin^2\theta}{\sin^2\theta+\cos^2\theta}=\frac{9}{10}$;

$$\therefore \sin\theta=\tfrac{3}{10}\sqrt{10},\ \cos\theta=\sqrt{1-\tfrac{9}{10}}=\tfrac{1}{10}\sqrt{10}.$$

For x put $x'\cos\theta-y'\sin\theta$, and for y put $x'\sin\theta+y'\cos\theta$ then $3xy-4x^2=a^2$ will become $\frac{1}{2}x^2-\frac{9}{2}y^2=a^2$,

$$\text{or, } x^2-9y^2=2a^2.$$

2. Let $x'=x\cos\theta-y\sin\theta$, and $y'=x\sin\theta+y\cos\theta$,

$$\text{or } x'=\tfrac{1}{2}\sqrt{2}(x-y), \text{ and } y'=\tfrac{1}{2}\sqrt{2}(x+y);$$

$$\therefore x'y'=\tfrac{1}{2}(x^2-y^2)=\tfrac{1}{2}a^2.$$

3. Here $A=4$, $B=12$, $C=9$; $\therefore \tan 2\theta=-\frac{12}{5}$,

$$\therefore \tan\theta=\tfrac{3}{2},\ \sin\theta=\frac{3}{\sqrt{13}},\ \cos\theta=\frac{2}{\sqrt{13}};$$

$$\text{hence } x'^2=\tfrac{13}{100}, \text{ or } x'=\pm\tfrac{1}{10}\sqrt{13};$$

the equation therefore represents two parallel straight lines, each inclined to the axis of x at an angle whose tangent is $-\frac{2}{3}$, and whose distance from each other is $\pm\frac{1}{10}\sqrt{13}$.

4. Here $A=1$, $B=1$, $C=1$; $\therefore \tan 2\theta=1\div 0=\infty$, the tang. of $90°$; $\therefore \theta=45°$, $\sin\theta=\cos\theta=\frac{1}{2}\sqrt{2}$;

$$\text{hence } 3x'^2+y'^2=2, \text{ or } \frac{x^2}{\frac{2}{3}}+\frac{y^2}{2}=1,$$

the equation to an ellipse whose axes are $2a=\frac{2}{3}\sqrt{6}$, and $2b=2\sqrt{2}$.

5. Here we have $A=9$, $B=-30$, $C=25$, $D=21$, $E=-35$, $F=10$; and since $B^2-4A\cdot C=0$, we cannot get rid of the simple powers of x and y; but we have $\tan 2\theta=\frac{15}{8}$; $\therefore \tan\theta=\frac{3}{5}$, $\sin\theta=\dfrac{3}{\sqrt{34}}$, $\cos\theta=\dfrac{5}{\sqrt{34}}$; and applying these in the usual way, the given equation is transformed to $34y'^2-7\sqrt{34}y'+10=0$, an equation yielding two numerical values of y'. So that the given equation represents two parallel straight lines. It will be found to be the product of $5y-3x-2=0$ and $5y-3x-5=0$, each of which represents a straight line the tangent of whose inclination to the axis of x is $\frac{3}{5}$.

6. Here we have $A=1$, $B=-6$, $C=1$, $D=-6$, $E=2$; and since $B\div(A-C)$ is $=\infty$, we can get rid of the simple powers of x and y; accordingly, we obtain $h=0$, $k=-1$; and putting $y=y'-1$, the given equation is transformed to

$x'^2-6x'y'+y'^2+4=0$. Now, $\tan 2\theta$ being $=\infty$, we have $\theta=45°$, $\sin\theta=\cos\theta=\frac{1}{2}\sqrt{2}$; and transforming we obtain

$$x''^2-2y''^2=2, \text{ or } \frac{x''^2}{2}-\frac{y''^2}{1}=1,$$

the equation to an hyperbola, whose axes are $2a=2\sqrt{2}$ and $2b=2$.

7. Here we have $x^2+2xy-y^2-2=0$, where $A=1$, $B=2$, $C=-1$, $F=-2$; $\tan 2\theta=1$; $\therefore \tan\theta=\sqrt{2}-1$, $\sin\theta=\frac{1}{2}\sqrt{2-\sqrt{2}}$, $\cos\theta=\frac{1}{2}\sqrt{2+\sqrt{2}}$. Accordingly, the equation will assume the form $x'^2-y'^2=\sqrt{2}$, the eqn. to a rectangular hyperbola, where $a^2=\sqrt{2}$, or $a=b=2^{\frac{1}{4}}$.

8. Here $B^2-4A\cdot C=0$. $\operatorname{Tan} 2\theta=-2\div 0=\infty$; $\therefore \theta=45°$, $\sin\theta=\cos\theta=\frac{1}{2}\sqrt{2}$. Accordingly, the eqn. will assume the form $\quad y'^2-2\sqrt{2}x'+2\sqrt{2}y'+8=0$,

where $C'=1$, $D'=-2\sqrt{2}$, $E'=2\sqrt{2}$, $F=8$.

$$\left(y'+\frac{E'}{2C'}\right)^2=-\frac{D'}{C'}\left(x'+\frac{F}{D'}-\frac{E'^2}{4C'\cdot D'}\right),$$

$$\text{or } (y'+\sqrt{2})=2\sqrt{2}(x'-1\tfrac{1}{2}\sqrt{2});$$

$$\therefore y''^2=2\sqrt{2}x'',$$

the equation to a parabola; the coordinates of the vertex being $1\frac{1}{2}\sqrt{2}$ and $-\sqrt{2}$.

9. Here we have $x^2-2xy+y^2-2x-2y-3=0$;

$$B^2-4A\cdot C=0. \quad \operatorname{Tan} 2\theta=\infty, \ \therefore \theta=45°,$$

$\sin\theta=\cos\theta=\frac{1}{2}\sqrt{2}$. Accordingly, the equation is transformed to $2y'^2-2\sqrt{2}x'-3=0$, where $C'=2$, $D'=-2\sqrt{2}$, $F=-3$; we hence obtain

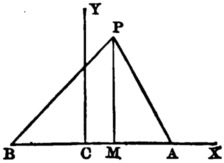
$$y'^2=\sqrt{2}\left(x'+\frac{3}{2\sqrt{2}}\right), =\sqrt{2}(x'+\tfrac{3}{4}\sqrt{2});$$

the eqn. to a parabola; the coordinates of the vertex being $-\frac{3}{4}\sqrt{2}$ and 0; latus rectum $=\sqrt{2}$.

10. Here $h=\frac{1}{2}$, $k=-\frac{1}{2}$; transformed equation, $3x'^2-2x'y'+y'^2-4\frac{1}{2}=0$; $\tan 2\theta=-1$, $\therefore 2\theta=135°, \theta=67\frac{1}{2}°$; $\sin\theta=\frac{1}{2}\sqrt{2+\sqrt{2}}$, $\cos\theta=\frac{1}{2}\sqrt{2-\sqrt{2}}$; whence, by further transformation,

$$(4-2\sqrt{2})x'^2+(4+2\sqrt{2})y'^2=9,$$

the eqn. to an ellipse; the coordinates of the centre $\frac{1}{2}$ and $-\frac{1}{2}$, and the axes, $a=1\frac{1}{2}\sqrt{2+\sqrt{2}}$, $b=1\frac{1}{2}\sqrt{2-\sqrt{2}}$.



11. Take C, the middle point of BA as origin; $BA=2n$, $CM=x$, $PM=y$;

$$\tan A=\frac{y}{n-x},\ \tan B=\frac{y}{n+x};$$

$$\tan(A-B)=\frac{\tan A-\tan B}{1+\tan A\tan B}=\frac{2xy}{n^2-x^2+y^2}=\tan 30°=\tfrac{1}{3}\sqrt{3};$$

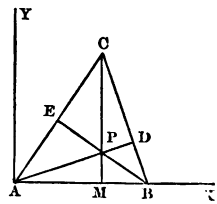
whence $x^2+2\sqrt{3}xy-y^2-n^2=0$.

$\tan 2\theta=\sqrt{3}=\tan 60°$, $\therefore\ \theta=30°$, $\sin\theta=\frac{1}{2}$, $\cos\theta=\frac{1}{2}\sqrt{3}$;

hence the transformed eqn. is $x'^2-y'^2=\frac{1}{2}n^2$;

or the locus of P is an equilateral hyperbola, whose axes are each $=\frac{1}{2}\sqrt{2}n$, and whose transverse axis makes with the base of the triangle an angle $=30°$.

12. Let ACB be one of the equal triangles on the base AB. Draw AD, BE perpendiculars to BC, AC; and through C draw the straight line CPM to meet AB.



Let A be the origin of coordinates, $AM=x'$, $CM=y'$; and put $AB=c$.

The eqn. to AC, passing through the points $(0, 0)$ and (x', y'), is

$$y=\frac{y'}{x'}x.$$

The eqn. to BC, passing through the points $(c, 0)$ and (x', y'), is
$$y=\frac{y'}{x'-c}(x-c).$$

Now, BE being perpend. to AC, and passing through the point $(c, 0)$, its eqn. is

$$y=-\frac{x'}{y'}(x-c).$$

The eqn. to AD, a perpend. on BC from the origin, is

$$y=-\frac{x'-c}{y'}x.$$

Now, at the point P where these perpendiculars intersect, the ordinate is common;

$$\therefore \frac{x'}{y'}(x-c)=\frac{x'-c}{y'}x;$$

whence $x=x'$, that is, x, the abscissa of the point P, is also the abscissa of the point C; $\therefore$ CPM is perpend. to AB.

Now, to find the locus of P for all triangles having the same base and altitude,

let $AM=x$, $PM=y$, $CM=p$, the constant altitude.

$$\cot ABD=\tan DAB=\frac{PM}{AM}=\frac{y}{x};$$

$$\tan ABD=\frac{CM}{BM}=\frac{p}{c-x};$$

by mult^n, $\quad 1=\dfrac{py}{cx-x^2}$, or $x^2+py-cx=0$;

where $C'=1$, $D'=p$, $E'=-c$.

$$\left(x'-\frac{c}{2}\right)^2=-p\left(y'-\frac{c^2}{4p}\right),$$

which is the equation to a parabola having its axis parallel to that of y, the coordinates of its vertex being $\frac{c}{2}$ and $\frac{c^2}{4p}$, and its latus rectum $=p$.

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